# Thermodynamics of the in-medium chiral condensate

S. Fiorilla, N. Kaiser and W. Weise
*Physik-Department, Technische Universität München, D-85747 Garching, Germany*

The temperature dependence of the chiral condensate in isospin-symmetric nuclear matter at varying baryon densities is investigated starting from a realistic free energy density of the correlated nuclear many-body system. The framework is thermal in-medium chiral effective field theory which permits a systematic calculation of the quark mass dependence of the free energy density. One- and two-pion exchange processes, virtual $\Delta(1232)$-isobar excitations and Pauli blocking corrections are treated up to and including three-loop order. It is found that nuclear matter remains in the Nambu-Goldstone phase with spontaneously broken chiral symmetry at least in the range of temperatures $T \lesssim 100\,\mathrm{MeV}$ and baryon densities up to about twice the density of normal nuclear matter.



PACS: 12.38.Bx, 21.65.+f
Keywords: In-medium chiral condensate, long-range correlations from one- and two-pion exchange in heated nuclear matter, QCD phase diagram.

The chiral condensate $\langle \bar{q}q \rangle$, i.e. the expectation value of the scalar quark density, plays a fundamental role as an order parameter of spontaneously broken chiral symmetry in the hadronic low-energy phase of QCD. The variation of $\langle \bar{q}q \rangle$ with temperature and baryon density is a key issue for locating the chiral transition boundary in the QCD phase diagram. The melting of the condensate at high temperatures and/or densities determines the crossover from the Nambu-Goldstone phase to the Wigner-Weyl realization of chiral symmetry in QCD.

It is thus of principal interest to perform a systematically organized calculation of the thermodynamics of the chiral condensate. Such a calculation requires knowledge of the free energy density as a function of the quark mass (or equivalently, as a function of the pion mass). The appropriate framework for such a task is in-medium chiral effective field theory with its explicit access to one- and two-pion exchange dynamics and the resulting two- and three-body correlations in the presence of a dense nuclear medium.

Previous studies of the in-medium variation of the chiral condensate were mostly concerned with the density dependence of $\langle \bar{q}q \rangle$ at zero temperature, using different approaches such as QCD sum rules [1] or models [2, 3] based on the boson exchange phenomenology of nuclear forces. Temperature effects have been included in schematic Nambu - Jona-Lasinio (NJL) approaches [4, 5]. Such NJL models work with quarks as quasiparticles and provide useful insights into dynamical mechanisms behind spontaneous chiral symmetry breaking and restoration, but they do not properly account for nucleons and their many-body correlations, a prerequisite for a more realistic treatment.

The present work extends a recent chiral effective field theory calculation [6] of the density-dependent in-medium $\langle \bar{q}q \rangle$ condensate to finite temperatures $T$. A related chiral effective field theory approach to nuclear matter at $T = 0$ performing resummations to all orders has been reported in ref.[7]. Corrections to the linear density approximation are obtained by differentiating the interaction parts of the free energy density of isospin-symmetric nuclear matter with respect to the (squared) pion mass. Effects from one-pion exchange (with $m_\pi$-dependent vertex corrections), iterated $1\pi$-exchange, and irreducible $2\pi$-exchange including intermediate $\Delta(1232)$-isobar excitations, with Pauli-blocking corrections up to three-loop order are systematically treated. The dominant nuclear matter effects on the dropping condensate are supplemented by a further small reduction due to interacting thermal pions. To anticipate the result: we find that the delayed tendency towards chiral symmetry restoration with increasing baryon density $\rho$, observed at $T = 0$ [6] in the same framework, gets gradually softened with increasing temperature. An approximately linear decrease of the quark condensate with increasing $\rho$ is recovered at temperatures around $T \simeq 100\,\mathrm{MeV}$. However, no rapid drive towards a first order chiral phase transition is seen, at least up to $\rho \lesssim 2\,\rho_0$ where $\rho_0 = 0.16\ \mathrm{fm}^{-3}$ is the density of normal nuclear matter.

Consider the free energy density $\mathcal{F} = \rho \bar{F}(\rho, T)$ of isospin-symmetric (spin-saturated) nuclear matter, with $\bar{F}(\rho, T)$ the free energy per particle. In the approach to nuclear matter based on in-medium chiral perturbation theory [8, 9] the free energy density is given by a sum of convolution integrals of the form,

$$\begin{aligned}
\rho\,\bar{F}(\rho,T) = 4\int_0^\infty & \mathrm{d}p\, p\, \mathcal{K}_1\, d(p) \\
& + \int_0^\infty \mathrm{d}p_1 \int_0^\infty \mathrm{d}p_2\, \mathcal{K}_2\, d(p_1) d(p_2) \\
& + \int_0^\infty \mathrm{d}p_1 \int_0^\infty \mathrm{d}p_2 \int_0^\infty \mathrm{d}p_3\, \mathcal{K}_3\, d(p_1) d(p_2) d(p_3) \\
& + \rho\, \bar{\mathcal{A}}(\rho, T)\,, \qquad (1)
\end{aligned}$$

where $\mathcal{K}_1, \mathcal{K}_2$ and $\mathcal{K}_3$ are one-body, two-body and three-body kernels, respectively. The last term, the so-called anomalous contribution $\bar{\mathcal{A}}(\rho, T)$ is a special feature at finite temperatures [10] with no counterpart in the calculation of the groundstate energy density at $T = 0$. As shown in ref.[8] the anomalous contribution arising in the present context from second-order pion exchange has actually very little influence on the equation of state of nuclear matter at moderate temperatures $T < 50\,\mathrm{MeV}$.

The quantity

$$d(p) = \frac{p}{2\pi^2}\left[1 + \exp\frac{p^2/2M_N - \tilde{\mu}}{T}\right]^{-1}, \qquad (2)$$

denotes the density of nucleon states in momentum space. It is the product of the temperature dependent Fermi-Dirac distribution and a kinematical prefactor $p/2\pi^2$ which has been included in $d(p)$ for convenience. $M_N$ stands for the (free) nucleon mass. The particle density $\rho$ is calculated as

$$\rho = 4\int_0^\infty \mathrm{d}p\, p\, d(p)\,. \qquad (3)$$

This relation determines the dependence of the effective one-body chemical potential $\tilde{\mu}(\rho, T; M_N)$ on the thermodynamical variables $(\rho, T)$ and indirectly also on the nucleon mass $M_N$. The one-body kernel $\mathcal{K}_1$ in eq.(1) provides the contribution of the non-interacting nucleon gas to the free energy density and it reads [8]

$$\mathcal{K}_1 = M_N + \tilde{\mu} - \frac{p^2}{3M_N} - \frac{p^4}{8M_N^3}\,. \qquad (4)$$

The first term in $\mathcal{K}_1$ gives the leading contribution (density $\rho$ times nucleon rest mass $M_N$) to the free energy density. The remaining terms account for (relativistically improved) kinetic energy corrections.

Our starting point is the Feynman-Hellmann theorem which relates the temperature dependent in-medium quark condensate $\langle\bar{q}q\rangle(\rho, T)$ to the quark mass derivative of the free energy density of isospin-symmetric (spin-saturated) nuclear matter. Using the Gell-Mann-Oakes-Renner relation $m_\pi^2 f_\pi^2 = -m_q\langle 0|\bar{q}q|0\rangle$ one finds for the ratio of the in-medium to vacuum quark condensate

$$\frac{\langle\bar{q}q\rangle(\rho, T)}{\langle 0|\bar{q}q|0\rangle} = 1 - \frac{\rho}{f_\pi^2}\frac{\partial\bar{F}(\rho, T)}{\partial m_\pi^2}\,, \qquad (5)$$

where the derivative with respect to $m_\pi^2$ is to be taken at fixed $\rho$ and $T$. The quantities $\langle 0|\bar{q}q|0\rangle$ (vacuum quark condensate) and $f_\pi$ (pion decay constant) are to be understood as taken in the chiral limit, $m_q \to 0$. Likewise, $m_\pi^2$ stands for the leading linear term in the quark mass expansion of the squared pion mass.

In the one-body kernel $\mathcal{K}_1$ the quark (or pion) mass dependence is implicit via its dependence on the nucleon mass $M_N$. The condition $\partial\rho/\partial M_N = 0$ applied to eq.(3) leads to the following dependence of the effective one-body chemical potential $\tilde{\mu}$ on the nucleon mass $M_N$:

$$\frac{\partial\tilde{\mu}}{\partial M_N} = \frac{3\rho}{2M_N\Omega_0''}\,, \qquad \Omega_0'' = -4M_N\int_0^\infty \mathrm{d}p\,\frac{d(p)}{p}\,. \qquad (6)$$

The nucleon sigma term $\sigma_N = \langle N|m_q\bar{q}q|N\rangle = m_\pi^2\,\partial M_N/\partial m_\pi^2$ measures the variation of the nucleon mass $M_N$ with the quark (or pion) mass. Combining both relationships leads to the $m_\pi^2$-derivative of the one-body kernel:

$$\frac{\partial\mathcal{K}_1}{\partial m_\pi^2} = \frac{\sigma_N}{m_\pi^2}\left\{1 + \frac{3\rho}{2M_N\Omega_0''} + \frac{p^2}{3M_N^2} + \frac{3p^4}{8M_N^4}\right\}. \qquad (7)$$

In the limit of zero temperature, $T = 0$, the terms in eq.(7) reproduce the linear decrease of the chiral condensate with density. The kinetic energy corrections account for the (small) difference between the scalar and the vector (i.e. baryon number) density. In the actual calculation we use the chiral expansion of the nucleon sigma term $\sigma_N$ to order $\mathcal{O}(m_\pi^4)$ as given in eq.(19) of ref.[6]. The empirical value of the nucleon sigma term (at the physical pion mass $m_\pi = 135\,\mathrm{MeV}$) is $\sigma_N = (45 \pm 8)\,\mathrm{MeV}$ [11]. In numerical calculations we choose the central value.

The two- and three-body kernels, $\mathcal{K}_2$ and $\mathcal{K}_3$, related to one-pion exchange and iterated one-pion exchange have already been given in explicit form in ref.[8]. Their derivatives with respect to the squared pion mass, $\partial\mathcal{K}_{2,3}/\partial m_\pi^2$, are hence obvious and do not need to be written out here. The same applies to the anomalous contribution $\bar{\mathcal{A}}(\rho, T)$ (see eqs.(14,15) in ref.[8]) and to the two- and three-body kernels related to $2\pi$-exchange with excitation of virtual $\Delta(1232)$-isobars (see section 6 in ref.[9]). In case of the one-pion exchange contribution we include the $m_\pi$-dependent vertex correction factor $\Gamma(m_\pi)$ as discussed in section 2.1 of ref.[6]. The short-distance contact term which produces a $T$-independent correction of order $\rho^2$ to the in-medium condensate is treated exactly in the same way as in ref.[6], i.e. all terms with a non-analytical quark-mass dependence generated by pion-loops are taken into account.

Let us now turn to some three-loop contributions which are new and of special relevance for the in-medium chiral condensate. The first one comes from the pion selfenergy diagram shown in Fig. 1. It gives rise to the following derivative of the two-body kernel:

$$\frac{\partial\mathcal{K}_2^{(\pi)}}{\partial m_\pi^2} = \frac{3g_A^2 m_\pi^2}{\pi^2(4f_\pi)^4}\Big\{\bar{\ell}_3\big[(X_{12}^-)^2 - (X_{12}^+)^2\big] + (4\bar{\ell}_3 - 1) \times \big(X_{12}^+ - X_{12}^-\big) + (2\bar{\ell}_3 - 1)\ln\frac{X_{12}^-}{X_{12}^+}\Big\}, \qquad (8)$$

with the abbreviations $X_{ij}^\pm = [1 + (p_i \pm p_j)^2/m_\pi^2]^{-1}$ and the $\pi\pi$ low-energy constant $\bar{\ell}_3 \simeq 3$.

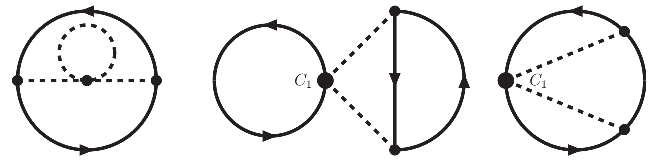


FIG. 1. Three-loop contributions to the free energy density of nuclear matter that are relevant for the in-medium chiral condensate. Left diagram: pion self-energy correction; central and right diagram: two-pion exchange Hartree and Fock terms involving the $\pi\pi NN$ contact interaction proportional to the low-energy constant $c_1 m_\pi^2$.

The chiral $\pi\pi NN$ contact vertex proportional to $c_1 m_\pi^2$ generates $2\pi$-exchange Hartree and Fock diagrams, also shown in Fig.1. Concerning the free energy density $\rho\bar{F}(\rho, T)$ or the equation of state of nuclear matter their contributions are actually almost negligible. However, when taking the derivative with respect to $m_\pi^2$ as required for the calculation of the in-medium condensate, these contributions turn out to be of similar importance as other interaction terms. The corresponding contribution to the derivative of the two-body kernel reads:

$$\frac{\partial \mathcal{K}_2^{(c_1)}}{\partial m_\pi^2} = \frac{g_A^2 c_1 m_\pi^3}{8\pi f_\pi^4}\left\{G\left(\frac{p_1+p_2}{2m_\pi}\right) - G\left(\frac{p_1-p_2}{2m_\pi}\right)\right\}, \quad (9)$$

with the auxiliary function:

$$G(x) = 8x(3+x^2)\arctan x - 5\ln(1+x^2) - 100x^2\,. \quad (10)$$

The $2\pi$-exchange Hartree diagram with one $c_1 m_\pi^2$-vertex contributes the following piece to the three-body kernel:

$$\frac{\partial \mathcal{K}_3^{(c_1)}}{\partial m_\pi^2} = \frac{6 g_A^2 c_1 p_3}{f_\pi^4}\left\{(X_{12}^+ - X_{12}^-)(X_{12}^+ + X_{12}^- - 3) + \ln\frac{X_{12}^+}{X_{12}^-}\right\}, \quad (11)$$

while the three-body term associated with the $2\pi$-exchange Fock diagram with one $c_1 m_\pi^2$-vertex gives:

$$\frac{\partial \mathcal{K}_3^{(c_1)}}{\partial m_\pi^2} = \frac{3 g_A^2 c_1}{f_\pi^4}\left[\frac{p_2}{p_3} + \frac{p_3^2 - p_2^2 - m_\pi^2}{4p_3^2}\ln\frac{X_{23}^-}{X_{23}^+}\right] \times\left[p_1 + \frac{p_3^2 - p_1^2 - 3m_\pi^2}{4p_3}\ln\frac{X_{13}^-}{X_{13}^+} + (p_1+p_3)X_{13}^+ + (p_1-p_3)X_{13}^-\right]. \quad (12)$$

Last not least we incorporate the effects of thermal pions. Through its $m_\pi^2$-derivative the pressure (or free energy density) of thermal pions gives rise to a further reduction of the $T$-dependent in-medium condensate. In the two-loop approximation of chiral perturbation theory including effects from the $\pi\pi$-interaction one finds the following shift of the condensate ratio in the presence of the pionic heat bath [12–14]:

$$\frac{\delta\langle\bar{q}q\rangle(T)}{\langle 0|\bar{q}q|0\rangle} = -\frac{3m_\pi^2}{(2\pi f_\pi)^2}H_3\left(\frac{m_\pi}{T}\right)\left\{1 + \frac{m_\pi^2}{8\pi^2 f_\pi^2}\times\left[H_3\left(\frac{m_\pi}{T}\right) - H_1\left(\frac{m_\pi}{T}\right) + \frac{2-3\,\bar{\ell}_3}{8}\right]\right\}, \quad (13)$$

with the functions $H_{1,3}(m_\pi/T)$ defined by integrals over the Bose distribution of thermal pions:

$$H_1(y) = \int_y^\infty dx\,\frac{1}{\sqrt{x^2-y^2}\,(e^x-1)}\,,$$
$$H_3(y) = y^{-2}\int_y^\infty dx\,\frac{\sqrt{x^2-y^2}}{e^x-1}\,. \quad (14)$$

We proceed with a presentation of results. As input we consistently use the same parameters in the chiral limit as in our previous works [6], namely: $f_\pi = 86.5\,\mathrm{MeV}$, $g_A = 1.224$, $c_1 = -0.93\,\mathrm{GeV}^{-1}$ and $M_N = 882\,\mathrm{MeV}$. Concerning the contact term representing unresolved short-distance dynamics, we recall from ref. [6] that its quark mass dependence, estimated from recent lattice QCD results for the nucleon-nucleon potential [15], is negligibly small compared to that of the intermediate and long range (pion-exchange driven) pieces.

It is worth pointing out that in-medium chiral perturbation theory with this input produces a realistic nuclear matter equation of state [9], including a proper description of the thermodynamics of the (first-order) liquid-gas phase transition. Apart from temperature $T$, the additional "small" parameter in this approach is the nucleon Fermi momentum $p_F$ in comparison with the chiral scale, $4\pi f_\pi \sim 1\,\mathrm{GeV}$. Our three-loop calculation of the free energy density is reliable up to about twice the density of normal nuclear matter. It can be trusted over a temperature range (up to $T \sim 100$ MeV) in which the hot and dense matter still remains well inside the phase of spontaneously broken chiral symmetry.

Fig. 2 shows a representative example, at $T = 100$ MeV, displaying stepwise the effects of interaction contributions to the density dependence of $\langle\bar{q}q\rangle(\rho, T)$ from the chiral two- and three-body kernels $\mathcal{K}_{2,3}$. As in the $T = 0$ case studied previously [6], the pion-mass dependence of correlations involving virtual $\Delta(1232)$-excitations turns out to be specifically important in delaying the tendency towards chiral symmetry restoration as the density in-

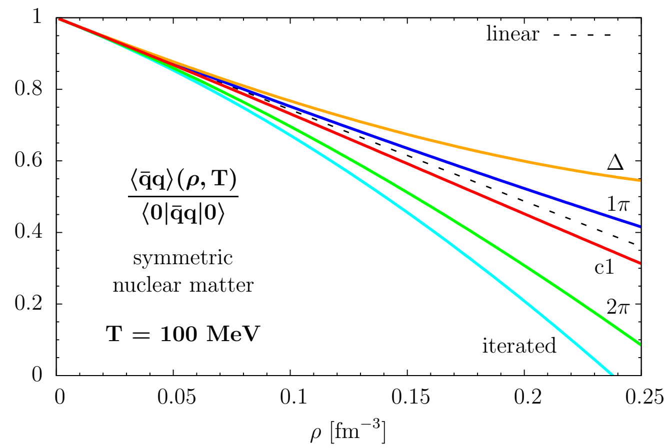


FIG. 2. Density dependence of the chiral condensate in isospin-symmetric nuclear matter taken at a temperature $T = 100$ MeV. Starting from the linear density dependence (dashed curve) characteristic of the free nucleon Fermi gas, the following interaction contributions are successively added: one-pion exchange Fock terms ($1\pi$), second order (iterated) pion exchange, irreducible two-pion exchange ($2\pi$), two- and three-body contributions from $2\pi$ exchange with intermediate $\Delta(1232)$-excitations ($\Delta$), and two-pion exchange with $\pi\pi NN$ vertex involving the chiral low-energy constant $c_1 m_\pi^2$. Pauli blocking effects are included throughout.

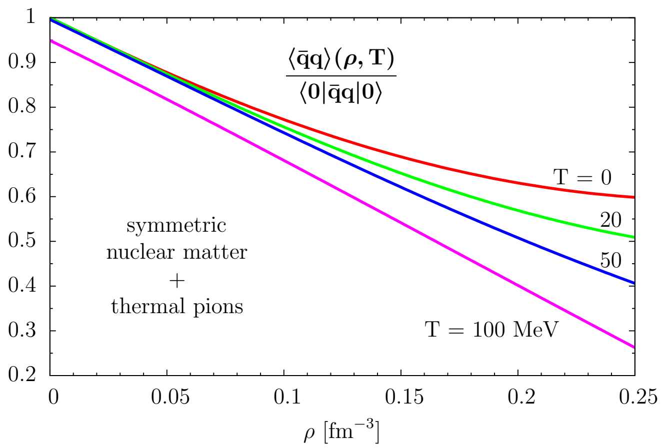


FIG. 3. Ratio of chiral condensate relative to its vacuum value as function of baryon density $\rho$ in symmetric nuclear matter, for different temperatures up to $T = 100$ MeV as indicated. The reduction effect due to thermal pions is included.

creases. Once all one- and two-pion exchange processes contributing to $\mathcal{K}_2$ and $\mathcal{K}_3$ are added up, the chiral condensate at $T = 100$ MeV recovers the linear density dependence characteristic of a free Fermi gas. However, this recovery is the result of a subtle balance between attractive and repulsive correlations and their detailed pion-mass dependences. Had we taken into account only iterated one-pion and irreducible two-pion exchanges, the system would have become instable not far above normal nuclear matter density as seen in Fig. 2. In fact, this instability would have appeared at even much lower densities in the chiral limit ($m_\pi \to 0$). This emphasizes once more not only the importance of terms involving $\Delta(1232)$-excitations, but also the significance of explicit chiral symmetry breaking by small but non-zero quark masses in QCD and the resulting physical pion mass, in governing nuclear scales.

Fig. 3 shows the systematics in the variation of the chiral condensate with temperature $T$ and baryon density $\rho$. These results include all nuclear correlation effects and also the (small) additional shift arising from thermal pions. The latter correction is visible only at the highest temperature considered here ($T = 100$ MeV) where the chiral condensate at zero density begins to deviate from its vacuum value. According to recent QCD lattice simulations the actual crossover transition at which the quark condensate $\langle \bar{q}q \rangle$ drops continuously to zero occurs around a temperature of $T_c \sim 170$ MeV [16].

At zero temperature, the hindrance of the dropping condensate at densities beyond normal nuclear matter comes primarily from three-body correlations through $\mathcal{K}_3$ which grow rapidly and faster than $\mathcal{K}_2$ as the density increases. The heating of the system reduces the influence of $\mathcal{K}_3$ relative to $\mathcal{K}_2$ continuously as the temperature rises, so that their balance at $T = 100$ MeV produces a small net effect in comparison with the free Fermi gas.

In summary, this is the first calculation of the quark condensate at finite temperature and density that systematically incorporates chiral two-pion exchange interactions in the nuclear medium. Correlations involving intermediate $\Delta(1232)$-excitations (i.e. the strong spin-isospin polarizability of the nucleon) together with Pauli blocking effects are demonstrated to play a crucial role in stabilizing the condensate at densities beyond that of equilibrated nuclear matter. The results reported here set important nuclear physics constraints for the QCD equation of state at baryon densities and temperatures that are of interest e.g. in relativistic heavy-ion collisions. In particular, we find no indication of a first order chiral phase transition at temperatures $T \lesssim 100$ MeV and baryon densities up to about twice the density of normal nuclear matter.


[1] E.G. Drukarev and E.M. Levin, *Nucl. Phys.* **A511**, 679 (1990); E.G. Drukarev, M.G. Ryskin, and V.A. Sadovnikova, *Eur. Phys. J.* **A4**, 171 (1999); T.D. Cohen, R.J. Furnstahl, and D.K. Griegel, *Phys. Rev.* **C45**, 1881(1992).

[2] G.Q. Li and C.M. Ko, *Phys. Lett.* **B338**, 118 (1994).

[3] R. Brockmann and W. Weise, *Phys. Lett.* **B367**, 40 (1996).

[4] T. Hatsuda and T. Kunihiro, *Phys. Reports* **247**, 221 (1994).

[5] S. Klimt, M. Lutz, and W. Weise *Phys. Lett.* **B249**, 386 (1990); M. Lutz, S. Klimt, and W. Weise, *Nucl. Phys.* **A542**, 521 (1992).

[6] N. Kaiser, P. de Homont, and W. Weise, *Phys. Rev.* **C77**, 025204 (2008); N. Kaiser and W. Weise, *Phys. Lett.* **B671**, 25 (2009).

[7] A. Lacour, J.A. Oller, and Ulf-G. Meißner, *J. Phys. G: Nucl. Part. Phys.* **37**, 125002 (2010); *Ann. Phys. (NY)* **326**, 241 (2011).

[8] S. Fritsch, N. Kaiser, and W. Weise, *Phys. Lett.* **B545**, 73 (2002).

[9] S. Fritsch, N. Kaiser, and W. Weise, *Nucl. Phys.* **A750**, 259 (2005).

[10] W. Kohn and J.M. Luttinger, *Phys. Rev.* **118**, 41 (1960); J.M. Luttinger and J.C. Ward, *Phys. Rev.* **118**, 1417 (1960).

[11] J. Gasser, H. Leutwyler, and M.E. Sainio, *Phys. Lett.* **B253**, 252 (1991).

[12] P. Gerber and H. Leutwyler, *Nucl. Phys.* **B321**, 387 (1989).

[13] D. Toublan, *Phys. Rev.* **D56**, 5629 (1997).

[14] N. Kaiser, *Phys. Rev.* **C59**, 2945 (1999).

[15] N. Ishii, S. Aoki, and T. Hatsuda, *Phys. Rev. Lett.* **99**, 022001 (2007); S. Aoki, T. Hatsuda, and N. Ishii, *Prog. Theor. Phys.* **123**, 89 (2010).

[16] A. Bazavov et al., *Phys. Rev.* **D80**, 014504 (2009); S. Borsanyi et al., *JHEP* **09**, 073 (2010);